IMA Journal of Numerical Analysis (2017) **37**, 1041–1065
doi: 10.1093/imanum/drw023
Advance Access publication on June 2, 2016

Mean-square convergence of a symplectic local discontinuous Galerkin method applied to stochastic linear Schrödinger equation

CHUCHU CHEN AND JIALIN HONG
State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
chenchuchu@lsec.cc.ac.cn hjl@lsec.cc.ac.cn

AND

LIHAI JI*
Institute of Applied Physics and Computational Mathematics, Beijing 100094, China
*Corresponding author: jilihai@lsec.cc.ac.cn

[Received on 18 March 2015; revised on 11 October 2015]

In this article, we investigate the mean-square convergence of a novel symplectic local discontinuous Galerkin method in $\mathbb{L}^2$-norm for stochastic linear Schrödinger equation with multiplicative noise. It is shown that the mean-square error is bounded, not only by the temporal and spatial step sizes, but also by their ratio. The mean-square convergence rate with respect to the computational cost is derived under appropriate assumptions for initial data and noise. Meanwhile, we show that the method preserves the discrete charge conservation law, which implies an $\mathbb{L}^2$-stability.

Keywords: symplectic method; local discontinuous Galerkin method; stochastic linear Schrödinger equation; $\mathbb{L}^2$-stability; charge conservation law; mean-square convergence.

1. Introduction

In this article we consider the stochastic linear Schrödinger equation with multiplicative noise

$$i\mathrm{d}u - (\Delta u + Q(x)u)\,\mathrm{d}t = u \circ \mathrm{d}W, \qquad u(x,0) = u_0(x), \tag{1.1}$$

where $t \in [0,T]$, $x \in \mathcal{O} \subset \mathbb{R}^d$ and $Q \in \mathbb{H}^3(\mathcal{O})$. We employ the periodic boundary condition, and the $\circ$ in the last term in (1.1) means that the product is of Stratonovich type, so that (1.1) is conservative and the $L^2(\mathcal{O})$-norm of the solution is a constant almost surely (charge conservation law) (see De Bouard & Debussche, 2003), i.e.,

$$\int_{\mathcal{O}} |u(x,t)|^2\,\mathrm{d}x = \int_{\mathcal{O}} |u_0(x)|^2\,\mathrm{d}x.$$

The multiplicative noise has been introduced in the context of Scheibe aggregates (Bang *et al.*, 1994; Rasmussen *et al.*, 1995) and in the context of inhomogeneous media (Bass *et al.*, 1989; Elgin, 1993). Here W on $\mathbb{L}^2(\mathcal{O})$ is a real-valued Wiener process with a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\in[0,T]})$. It has the expansion form $W(t,x,\omega) = \sum_{k=0}^{\infty} \beta_k(t,\omega)\phi e_k(x)$, with $(e_k)_{k\in\mathbb{N}^d}$ being an orthonormal basis of $\mathbb{L}^2(\mathcal{O})$, $\{\beta_k\}_{k\in\mathbb{N}^d}$ being a sequence of independent Brownian motions and $\phi \in \mathcal{L}_2(\mathbb{L}^2(\mathcal{O}); \mathbb{H}^\gamma(\mathcal{O}))$ being a

Hilbert–Schmidt operator. The phase flow of equation (1.1) is stochastic symplectic (see Chen & Hong, 2016), i.e.,

$$\bar{\omega}(t) = \int_{\mathcal{O}} \mathrm{d}(r(t)) \wedge \mathrm{d}(s(t))\, \mathrm{d}x = \bar{\omega}(0),$$

with r and s being the real and imaginary parts of u, respectively.

We propose a symplectic local discontinuous Galerkin method to equation (1.1) in order to, on one hand preserve the properties of the original problems as much as possible and, on another hand, combine the attractive properties of local discontinuous Galerkin method (see, e.g., Cockburn & Shu, 1998, 2001; Cockburn *et al.*, 2000). We refer interested readers to Xu & Shu (2005) and references therein for the numerical simulation of the deterministic Schrödinger equation based on local discontinuous Galerkin method, and to Antonopoulou & Plexousakis (2010) for the convergence analysis of discontinuous Galerkin method applied to the deterministic linear Schrödinger equation with time-variable domain. Because of the reason that equation (1.1) is meaningful in the sense of integral representation, we apply the midpoint scheme to discretize the temporal direction at first avoiding dealing with double temporal–spatial integrals, which is introduced by stochastic integral and local discontinuous Galerkin discretization. It is shown that the midpoint semidiscretization not only is a symplectic method, but also possesses the discrete charge conservation law. Furthermore, we show that the semidiscretization is of order 1 in mean-square convergence sense via a direct approach, whereas Chen & Hong (2016) proved the same result via a fundamental convergence theorem on the mean-square convergence for the temporal semidiscretizations. The main difficulty lies in the analysis of the mean-square convergence order for the spatial direction, where we use local discontinuous Galerkin method to discrete the semidiscretized equation and obtain the fully discrete method, which is called symplectic local discontinuous Galerkin method in this article. We solve it by means of the standard approximation theory of projection operator, Itô isometry and the adapted properties of processes u and W. As a result, we analyse the mean-square convergence error for the symplectic local discontinuous Galerkin method and derive the mean-square convergence rate with respect to the computational cost under appropriate hypothesis on initial data and noise. Moreover, theoretical analysis shows that the obtained fully discrete method is $\mathbb{L}^2$-stable and preserves the discrete charge conservation law.

The rest of this article is organized as follows. In Section 2, we propose the symplectic local discontinuous Galerkin method for stochastic Schrödinger equation and derive the discrete charge conservation law. In Section 3, we study the mean-square convergence of the obtained method and present the mean-square error estimation.

2. The symplectic local discontinuous Galerkin method

In this section, we will apply implicit midpoint scheme to (1.1) in the temporal direction, then we discretize the spatial direction by local discontinuous Galerkin method and obtain the fully discrete method.

2.1 *Temporal semidiscrete scheme*

The midpoint scheme for (1.1) reads

$$iu^{n+1} = iu^n - \Delta t\Big(\Delta u^{n+\frac{1}{2}} + Q(x)u^{n+\frac{1}{2}}\Big) + u^{n+\frac{1}{2}}\Delta \tilde{W}_n,\ n = 0, 1, \ldots, N, \tag{2.1}$$

where Δt is the time step size, $N = \frac{T}{\Delta t}$, $u^{n+\frac{1}{2}} = \frac{1}{2}(u^{n+1} + u^n)$, and $\Delta \tilde{W}_n = \sum_{k=0}^{\infty} \sqrt{\Delta t}\zeta_{k,n}^{\kappa}\phi e_k(x)$ with $\zeta_{k,n}^{\kappa}$ being the truncation of a $\mathcal{N}(0,1)$-distribution random variable $\xi_{k,n}$:

$$\zeta_{k,n}^{\kappa} = \begin{cases} \kappa & \text{if } \xi_{k,n} > \kappa; \\ \xi_n & \text{if } |\xi_{k,n}| \leq \kappa; \\ -\kappa & \text{if } \xi_{k,n} < -\kappa \end{cases}$$

with $\kappa := \sqrt{4|\ln(\Delta t)|}$. This choice is motivated by the fact that standard Gaussian random variables are unbounded for arbitrary values of Δt (for more details, see Milstein *et al.*, 2002). For the truncated Wiener process, we have the following properties:

$$\begin{aligned} &\text{(i)} \qquad \mathbb{E}\|\Delta \tilde{W}_n - \Delta W_n\|_{\mathbb{H}^1}^2 \leq K\Delta t^3, \\ &\text{(ii)} \qquad \mathbb{E}\|(\Delta \tilde{W}_n)^2 - (\Delta W_n)^2\|_{\mathbb{H}^1} \leq K\Delta t^2, \\ &\text{(iii)} \qquad \mathbb{E}\|(\Delta \tilde{W}_n)^2 - (\Delta W_n)^2\|_{\mathbb{H}^1}^2 \leq K\Delta t^3, \end{aligned} \tag{2.2}$$

where the constant K depends on $\|\phi\|_{\mathcal{L}_2(\mathbb{L}^2,\mathbb{H}^1)}$. Based on the fact that $\tilde{W}$ is real valued, by multiplying both sides of equation (2.1) by $\bar{u}^{n+\frac{1}{2}}$, which is the conjugate of $u^{n+\frac{1}{2}}$, and then taking the imaginary part and integrating it over the whole space domain, we can get the discrete charge conservation law as follows.

Proposition 2.1 Under the periodic boundary conditions, the semidiscrete scheme (2.1) of the system (1.1) has the discrete charge conservation law, i.e.,

$$\int_{\mathcal{O}} |u^{n+1}(x)|^2 \, \mathrm{d}x = \int_{\mathcal{O}} |u^n(x)|^2 \, \mathrm{d}x, \quad n = 0, 1, \ldots, N. \tag{2.3}$$

Furthermore, the semidiscrete scheme (2.1) preserves the stochastic symplectic structure (see Chen & Hong, 2016).

Proposition 2.2 The implicit midpoint scheme (2.1) for the system (1.1) is stochastic symplectic.

2.2 *Temporal–spatial fully discrete method*

In this subsection, we consider the local discontinuous Galerkin method for the system (2.6) in the spatial direction and obtain the fully discrete method. To this end, we introduce some spatial-gird notation for the case $d = 1$, $\mathcal{O} = [L_f, L_r]$ for simplicity. We denote the mesh by $\mathrm{I}_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $1 \leq j \leq J$, where $L_f = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = L_r$. Let $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $1 \leq j \leq J$ with $h = \max\limits_{1\leq j\leq J} \Delta x_j$ being the maximum mesh size. Assume the mesh is regular, namely there is a constant $c > 0$ independent of h such that $\Delta x_j \geq ch, \quad 1 \leq j \leq J$.

If we set $u(x,t) = r(x,t) + is(x,t)$, where $r,\ s$ are real-valued functions, we can separate (1.1) into the following form

$$\begin{aligned} \mathrm{d}r &= (s_{xx} + Q(x)s)\,\mathrm{d}t + s \circ \mathrm{d}W, \\ \mathrm{d}s &= -(r_{xx} + Q(x)r)\,\mathrm{d}t - r \circ \mathrm{d}W. \end{aligned} \tag{2.4}$$

Introducing two additional new variables, $p = s_x,\ q = r_x$, the equation (2.4) can be rewritten as

$$\begin{aligned} \mathrm{d}r &= (p_x + Q(x)s)\ \mathrm{d}t + s \circ \mathrm{d}W, \\ p &= s_x, \\ \mathrm{d}s &= -(q_x + Q(x)r)\ \mathrm{d}t - r \circ \mathrm{d}W, \\ q &= r_x. \end{aligned} \tag{2.5}$$

We apply the midpoint scheme in the temporal direction of (2.5) to obtain the following first-order semidiscrete system

$$\begin{aligned} r^{n+1} &= r^n + \Big((p_x)^{n+\frac{1}{2}} + Q(x)s^{n+\frac{1}{2}}\Big)\Delta t + s^{n+\frac{1}{2}}\Delta \tilde{W}_n, \\ p^{n+\frac{1}{2}} &= (s_x)^{n+\frac{1}{2}}, \\ s^{n+1} &= s^n - \Big((q_x)^{n+\frac{1}{2}} + Q(x)r^{n+\frac{1}{2}}\Big)\Delta t - r^{n+\frac{1}{2}}\Delta \tilde{W}_n, \\ q^{n+\frac{1}{2}} &= (r_x)^{n+\frac{1}{2}}. \end{aligned} \tag{2.6}$$

We consider the local discontinuous Galerkin method for the system (2.6) in the spatial direction and obtain the fully discrete method: find $r_h, p_h, s_h, q_h \in V_h^k$, which now denote real piecewise polynomial of degree at most k, such that for all test functions $\nu_h, \omega_h, \alpha_h, \beta_h \in V_h^k = \{\nu : \nu \in P^k(\mathrm{I}_j);\ 1 \le j \le J\}$ with $P^k(\mathrm{I}_j)$ being the set of polynomials of degree up to k defined on the cell I_j.

$$\begin{aligned} &\int_{\mathrm{I}_j} r_h^{n+1}\nu_h\ \mathrm{d}x - \int_{\mathrm{I}_j} r_h^n \nu_h\ \mathrm{d}x - \Delta t\Big[(\hat{p}^{n+\frac{1}{2}}\nu_h^-)_{j+\frac{1}{2}} - (\hat{p}^{n+\frac{1}{2}}\nu_h^+)_{j-\frac{1}{2}}\Big] \\ &\quad + \Delta t \int_{\mathrm{I}_j} \Big(p_h^{n+\frac{1}{2}}(\nu_h)_x - s_h^{n+\frac{1}{2}} Q_h \nu_h\Big)\ \mathrm{d}x - \int_{\mathrm{I}_j} s_h^{n+\frac{1}{2}}\nu_h \Delta \tilde{W}_n\ \mathrm{d}x = 0, \\ &\int_{\mathrm{I}_j} p_h^{n+\frac{1}{2}}\omega_h\ \mathrm{d}x + \int_{\mathrm{I}_j} s_h^{n+\frac{1}{2}}(\omega_h)_x\ \mathrm{d}x - \Big[(\hat{s}^{n+\frac{1}{2}}\omega_h^-)_{j+\frac{1}{2}} - (\hat{s}^{n+\frac{1}{2}}\omega_h^+)_{j-\frac{1}{2}}\Big] = 0, \\ &\int_{\mathrm{I}_j} s_h^{n+1}\alpha_h\ \mathrm{d}x - \int_{\mathrm{I}_j} s_h^n \alpha_h\ \mathrm{d}x + \Delta t\Big[(\hat{q}^{n+\frac{1}{2}}\alpha_h^-)_{j+\frac{1}{2}} - (\hat{q}^{n+\frac{1}{2}}\alpha_h^+)_{j-\frac{1}{2}}\Big] \\ &\quad - \Delta t \int_{\mathrm{I}_j} \Big(q_h^{n+\frac{1}{2}}(\alpha_h)_x - r_h^{n+\frac{1}{2}} Q_h \alpha_h\Big)\ \mathrm{d}x + \int_{\mathrm{I}_j} r_h^{n+\frac{1}{2}}\alpha_h \Delta \tilde{W}_n\ \mathrm{d}x = 0, \\ &\int_{\mathrm{I}_j} q_h^{n+\frac{1}{2}}\beta_h\ \mathrm{d}x + \int_{\mathrm{I}_j} r_h^{n+\frac{1}{2}}(\beta_h)_x\ \mathrm{d}x - \Big[(\hat{r}^{n+\frac{1}{2}}\beta_h^-)_{j+\frac{1}{2}} - (\hat{r}^{n+\frac{1}{2}}\beta_h^+)_{j-\frac{1}{2}}\Big] = 0. \end{aligned} \tag{2.7}$$

In the sequel, we denote by $(u_h)^+_{j+\frac{1}{2}}$ and $(u_h)^-_{j+\frac{1}{2}}$ the values of u_h at $x_{j+\frac{1}{2}}$, from the right cell I_{j+1}, and from the left cell I_j, respectively. Also the numerical fluxes are of the general form

$$\hat{p}^{n+\frac{1}{2}} = (p^{n+\frac{1}{2}})^+,\ \hat{r}^{n+\frac{1}{2}} = (r^{n+\frac{1}{2}})^-,\ \hat{q}^{n+\frac{1}{2}} = (q^{n+\frac{1}{2}})^+,\ \hat{s}^{n+\frac{1}{2}} = (s^{n+\frac{1}{2}})^-, \tag{2.8}$$

where we have omitted the half-integer indices $j+\frac{1}{2}$ or $j-\frac{1}{2}$ as all quantities in (2.8) are computed at the same points.

REMARK 2.3 The choice for the fluxes (2.8) is not unique. The important point is that $\hat{r}$ and $\hat{q}$, $\hat{s}$ and $\hat{p}$ should be chosen from different directions.

With such a choice of fluxes (2.8), we can get the first main result about discrete charge conservation law of the symplectic local discontinuous Galerkin method (2.7).

THEOREM 2.4 Under the periodic boundary conditions, the symplectic local discontinuous Galerkin method (2.7) has the discrete charge conservation law, i.e.,

$$\int_{L_f}^{L_r} |u_h^{n+1}|^2 \, \mathrm{d}x = \int_{L_f}^{L_r} |u_h^n|^2 \, \mathrm{d}x, \quad n = 0, 1, 2, \ldots, N. \tag{2.9}$$

Proof. To complete the proof of the discrete charge conservation law. First, we write (2.7) using the notations $u_h^n = r_h^n + is_h^n$, $\psi_h^n = q_h^n + ip_h^n$ and take $\alpha_h = \nu_h$, $\beta_h = \omega_h$, then (2.7) becomes

$$\begin{aligned} & i\int_{\mathrm{I}_j} u_h^{n+1}\nu_h \, \mathrm{d}x - i\int_{\mathrm{I}_j} u_h^n \nu_h \, \mathrm{d}x - [(\hat{\psi}^{n+\frac{1}{2}}\nu_h^-)_{j+\frac{1}{2}} - (\hat{\psi}^{n+\frac{1}{2}}\nu_h^+)_{j-\frac{1}{2}}]\Delta t \\ & \quad + \Delta t\int_{\mathrm{I}_j} (\psi_h^{n+\frac{1}{2}}(\nu_h)_x - u_h^{n+\frac{1}{2}} Q_h \nu_h) \, \mathrm{d}x - \int_{\mathrm{I}_j} u_h^{n+\frac{1}{2}} \nu_h \Delta \tilde{W}_n \, \mathrm{d}x = 0, \\ & \int_{\mathrm{I}_j} \psi_h^{n+\frac{1}{2}} \omega_h \, \mathrm{d}x + \int_{\mathrm{I}_j} u_h^{n+\frac{1}{2}} (\omega_h)_x \, \mathrm{d}x - \left[(\hat{u}^{n+\frac{1}{2}}\omega_h^-)_{j+\frac{1}{2}} - (\hat{u}^{n+\frac{1}{2}}\omega_h^+)_{j-\frac{1}{2}}\right] = 0, \end{aligned} \tag{2.10}$$

where

$$\hat{u} = r_h^- + is_h^-, \quad \hat{\psi} = q_h^+ + ip_h^+. \tag{2.11}$$

We now take the complex conjugate for every terms in system (2.10)

$$\begin{aligned} & -i\int_{\mathrm{I}_j} \bar{u}_h^{n+1}\bar{\nu}_h \, \mathrm{d}x + i\int_{\mathrm{I}_j} \bar{u}_h^n \bar{\nu}_h \, \mathrm{d}x - \Delta t\left[(\bar{\hat{\psi}}^{n+\frac{1}{2}}\bar{\nu}_h^-)_{j+\frac{1}{2}} - (\bar{\hat{\psi}}^{n+\frac{1}{2}}\bar{\nu}_h^+)_{j-\frac{1}{2}}\right] \\ & + \Delta t\int_{\mathrm{I}_j} \left(\bar{\psi}_h^{n+\frac{1}{2}}(\bar{\nu}_h)_x - \bar{u}_h^{n+\frac{1}{2}} Q_h \bar{\nu}_h\right) \mathrm{d}x - \int_{\mathrm{I}_j} \bar{u}_h^{n+\frac{1}{2}} \bar{\nu}_h \Delta \tilde{W}_n \, \mathrm{d}x = 0, \\ & \int_{\mathrm{I}_j} \bar{\psi}_h^{n+\frac{1}{2}} \bar{\omega}_h \, \mathrm{d}x + \int_{\mathrm{I}_j} \bar{u}_h^{n+\frac{1}{2}} (\bar{\omega}_h)_x \, \mathrm{d}x - \left[(\bar{\hat{u}}^{n+\frac{1}{2}}\bar{\omega}_h^-)_{j+\frac{1}{2}} - (\bar{\hat{u}}^{n+\frac{1}{2}}\bar{\omega}_h^+)_{j-\frac{1}{2}}\right] = 0. \end{aligned} \tag{2.12}$$

We introduce a short-hand notation

$$\begin{aligned}\mathfrak{H}_j(u_h^n,\psi_h^n;v_h,\omega_h) &= i\int_{\mathrm{I}_j} u_h^{n+1}v_h\,\mathrm{d}x - i\int_{\mathrm{I}_j} u_h^n v_h\,\mathrm{d}x - \Delta t\int_{\mathrm{I}_j}\psi_h^{n+\frac{1}{2}}\omega_h\,\mathrm{d}x \\ &+ \Delta t\int_{\mathrm{I}_j}\left(\psi_h^{n+\frac{1}{2}}(v_h)_x - u_h^{n+\frac{1}{2}}Q_h v_h\right)\mathrm{d}x - \int_{\mathrm{I}_j} u_h^{n+\frac{1}{2}}v_h\Delta\tilde{W}_n\,\mathrm{d}x \\ &- \Delta t\int_{\mathrm{I}_j} u_h^{n+\frac{1}{2}}(\omega_h)_x\,\mathrm{d}x - \Delta t\Big[(\hat{\psi}^{n+\frac{1}{2}}v_h^-)_{j+\frac{1}{2}} - (\hat{\psi}^{n+\frac{1}{2}}v_h^+)_{j-\frac{1}{2}}\Big] \\ &+ \Delta t\Big[(\hat{u}^{n+\frac{1}{2}}\omega_h^-)_{j+\frac{1}{2}} - (\hat{u}^{n+\frac{1}{2}}\omega_h^+)_{j-\frac{1}{2}}\Big]. \end{aligned} \tag{2.13}$$

Then from (2.12), we also have the expression of $\bar{\mathfrak{H}}_j(u_h^n,\psi_h^n;v_h,\omega_h)$. If we take $v_h=\bar{u}_h^{n+\frac{1}{2}},\ \omega_h=\bar{\psi}_h^{n+\frac{1}{2}}$ in both functions $\mathfrak{H}_j(u_h^n,\psi_h^n;v_h,\omega_h)$ and $\bar{\mathfrak{H}}_j(u_h^n,\psi_h^n;v_h,\omega_h)$, both functions are zero. Hence, we obtain

$$\mathfrak{H}_j(u_h^n,\psi_h^n;\bar{u}_h^{n+\frac{1}{2}},\bar{\psi}_h^{n+\frac{1}{2}}) - \bar{\mathfrak{H}}_j(u_h^n,p_h^n;\bar{u}_h^{n+\frac{1}{2}},\bar{\psi}_h^{n+\frac{1}{2}}) = 0. \tag{2.14}$$

By the relation (2.11) for the numerical fluxes, (2.14) becomes

$$\begin{aligned} &\mathrm{i}\int_{\mathrm{I}_j}\left(|u_h^{n+1}|^2-|u_h^n|^2\right)\mathrm{d}x + \underbrace{\Delta t\int_{\mathrm{I}_j}\left(\psi_h^{n+\frac{1}{2}}(\bar{u}_h^{n+\frac{1}{2}})_x + \bar{u}_h^{n+\frac{1}{2}}(\psi_h^{n+\frac{1}{2}})_x\right)\mathrm{d}x}_{A} \\ &- \underbrace{\Delta t\int_{\mathrm{I}_j}\left(\psi_h^{\star n+\frac{1}{2}}(u_h^{n+\frac{1}{2}})_x + u_h^{n+\frac{1}{2}}(\bar{\psi}_h^{n+\frac{1}{2}})_x\right)\mathrm{d}x}_{B} - \underbrace{\Delta t\Big[(\psi_h^+\bar{u}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\bar{\psi}_h^+u_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}}\Big]}_{C} \\ &+ \underbrace{\Delta t\Big[(u_h^-\bar{\psi}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\bar{u}_h^-\psi_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}}\Big]}_{D} + \underbrace{\Delta t\Big[(\psi_h^+\bar{u}_h^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}} - (\bar{\psi}_h^+u_h^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}\Big]}_{G} \\ &- \underbrace{\Delta t\Big[(u_h^-\bar{\psi}_h^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}} - (\bar{u}_h^-\psi_h^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}\Big]}_{H} = 0, \end{aligned} \tag{2.15}$$

where $(\psi_h^+\bar{u}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} = (\psi_h^{n+\frac{1}{2},+}\bar{u}_h^{n+\frac{1}{2},-})_{j+\frac{1}{2}}$.

By Leibniz formula for derivatives, we can derive

$$\begin{aligned} A &= \Delta t\int_{\mathrm{I}_j}(\psi_h^{n+\frac{1}{2}}\bar{u}_h^{n+\frac{1}{2}})_x\,\mathrm{d}x = \Delta t\Big[(\psi_h^-\bar{u}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\psi_h^+\bar{u}_h^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}\Big], \\ B &= \Delta t\int_{\mathrm{I}_j}(\bar{\psi}_h^{n+\frac{1}{2}}u_h^{n+\frac{1}{2}})_x\,\mathrm{d}x = \Delta t\Big[(u_h^-\bar{\psi}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (u_h^+\bar{\psi}_h^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}\Big] \end{aligned}$$

and then

$$A - B = 2\mathrm{i}\Delta t\Big[\mathrm{Im}(\psi_h^-\bar{u}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \mathrm{Im}(\psi_h^+\bar{u}_h^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}\Big]. \tag{2.16}$$

Using $a - \bar{a} = 2i\,\mathrm{Im}(a)$, for $a \in \mathbb{C}$, we have

$$C = 2\mathrm{i}\Delta t \mathrm{Im}(\psi_h^+ \bar{u}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}}, \quad D = -2\mathrm{i}\Delta t \mathrm{Im}(\psi_h^- \bar{u}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}},$$
$$G = 2\mathrm{i}\Delta t \mathrm{Im}(\psi_h^+ \bar{u}_h^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}, \quad H = -2\mathrm{i}\Delta t \mathrm{Im}(\psi_h^+ \bar{u}_h^-)_{j-\frac{1}{2}}^{n+\frac{1}{2}}. \tag{2.17}$$

We combine all these equalities (2.15), (2.16) and (2.17) to obtain

$$\int_{\mathrm{I}_j} (|u_h^{n+1}|^2 - |u_h|^n)\,\mathrm{d}x + \hat{\Phi}_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{\Phi}_{j-\frac{1}{2}}^{n+\frac{1}{2}} = 0,$$

where the numerical entropy flux is given by

$$\hat{\Phi}_{j+\frac{1}{2}}^{n+\frac{1}{2}} = -2\Delta t \mathrm{Im}(\psi_h^+ \bar{u}_h^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}}, \quad \hat{\Phi}_{j-\frac{1}{2}}^{n+\frac{1}{2}} = -2\Delta t \mathrm{Im}(\psi_h^+ \bar{u}_h^-)_{j-\frac{1}{2}}^{n+\frac{1}{2}}.$$

Summing up over j, the flux terms vanish because of the periodic boundary conditions. Thus, we finish the proof. □

COROLLARY 2.5 The discrete charge conservation law trivially implies an L^2-stability of the numerical solution.

3. Error estimates for the fully discrete method

In this section, we will state the error estimate of the symplectic local discontinuous Galerkin method for the problem (1.1) with $d = 1$. In the sequel, $\mathbb{E}$ denotes an expectation operator of a random variable, and K, C are positive constants depending on coefficient Q, the finial time T and the initial data u_0, but independent of h and Δt. They may change from line to line.

In order to obtain the error estimate to the symplectic local discontinuous Galerkin method (2.7) with the fluxes (2.8), we divide the error into two parts:

$$u(t_n) - u_h^n = \underbrace{u(\cdot, t_n) - u^n}_{\text{Temporal error}} + \underbrace{u^n - u_h^n}_{\text{Spatial error}}. \tag{3.1}$$

3.1 *Temporal error*

To obtain the temporal error estimate, we need some regularity results of the numerical solution $u^n(x)$ for (2.6). We state it in the following two lemmas.

LEMMA 3.1 Assume that $Q \in \mathbb{H}^\gamma$ and $\mathbb{E}\|u^0\|_{\mathbb{H}^\gamma}^{2p} < \infty,\ \gamma = 0, 1, \cdots$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2; \mathbb{H}^\gamma)$. We have the following regularity of temporal semidiscretization, i.e., for $p \geq 1$, there exists a constant $K \equiv K(p)$ such that

$$\mathbb{E}\|u^n\|_{\mathbb{H}^\gamma}^{2p} \leq K, \quad n = 1, 2, \ldots, N. \tag{3.2}$$

Proof. First, we rewrite temporal semidiscretization system (2.6) into the function of u^n:

$$u^{n+1} = \hat{S}_{\Delta t} u^n - i\Delta t T_{\Delta t} Q u^{n+\frac{1}{2}} - i T_{\Delta t} u^{n+\frac{1}{2}} \Delta \tilde{W}_n, \tag{3.3}$$

where u^n denotes the complex function $r^n + is^n$, operators are defined by $\hat{S}_{\Delta t} = (\mathrm{I} + i\frac{\Delta t}{2}\partial_{xx})^{-1}(\mathrm{I} - i\frac{\Delta t}{2}\partial_{xx})$ and $T_{\Delta t} = (\mathrm{I} + i\frac{\Delta t}{2}\partial_{xx})^{-1}$, where I is an identity operator.

In particular, $T_{\Delta t}$ is a bounded linear inverse operator from $\mathbb{L}^2$ to $\mathbb{L}^2$. Let $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ and $\{e_k\}_{k\in\mathbb{N}} \subset \mathbb{L}^2$ be the eigenvalues and eigenfunctions of the linear operator ∂_{xx}. The corresponding eigenvalues of $\mathrm{I}+i\frac{\Delta t}{2}\partial_{xx}$ are $\{1 + i\frac{\Delta t}{2}\lambda_k\}_{k\in\mathbb{N}}$. Thus, the linear operator $T_{\Delta t}$ is well defined. Furthermore, it is easy to check that the operator $\|T_{\Delta t}\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)} \leq 1$ and $\hat{S}_{\Delta t}$ is isometry in $\mathbb{L}^2$, i.e., $\|\hat{S}_{\Delta t}\|_{\mathcal{L}(\mathbb{L}^2;\mathbb{L}^2)} = 1$. See De Bouard & Debussche (2006), for example.

Next, we replace the function of u^n into equation (3.3) iteratively. We obtain

$$u^n = \hat{S}^n_{\Delta t} u^0 - \mathrm{i}\Delta t \sum_{\ell=1}^{n} \hat{S}^{n-\ell}_{\Delta t} T_{\Delta t} Q u^{\ell-\frac{1}{2}} - \mathrm{i} \sum_{\ell=1}^{n} \hat{S}^{n-\ell}_{\Delta t} T_{\Delta t} u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{\ell-1}. \tag{3.4}$$

In order to bound function u^n, we insert the equality $u^{\ell-\frac{1}{2}} = \frac{1}{2}(\hat{S}_{\Delta t} + \mathrm{I})u^{\ell-1} + \frac{1}{2}\big(u^\ell - \hat{S}_{\Delta t}u^{\ell-1}\big)$ into the stochastic term and take $\mathbb{H}^\gamma$-norm to get

$$\begin{aligned} \|u^n\|^{2p}_{\mathbb{H}^\gamma} \leq{}& K\|u^0\|^{2p}_{\mathbb{H}^\gamma} + K\Delta t \sum_{\ell=1}^{n} \|u^{\ell-\frac{1}{2}}\|^{2p}_{\mathbb{H}^\gamma} \\ &+ K \left\| \frac{\mathrm{i}}{2} \sum_{\ell=1}^{n} \hat{S}^{n-\ell}_{\Delta t} T_{\Delta t}\big(\hat{S}_{\Delta t} + \mathrm{I}\big) u^{\ell-1} \Delta \tilde{W}_{\ell-1} \right\|^{2p}_{\mathbb{H}^\gamma} \\ &+ K n^{2p-1} \sum_{\ell=1}^{n} \big\|(u^\ell - \hat{S}_{\Delta t} u^{\ell-1}) \Delta \tilde{W}_{\ell-1}\big\|^{2p}_{\mathbb{H}^\gamma}. \end{aligned} \tag{3.5}$$

For the third term on the right-hand side of (3.5), using the fact that $u^{\ell-1}$ is independent of increment $\Delta \tilde{W}_{\ell-1}$ and Burkholder–Davis–Gundy-type inequality (for instance, see Proposition 9 in Appendix A.1 of Jentzen & Kloeden (2011)) we have

$$\begin{aligned} &\mathbb{E}\left\| \frac{\mathrm{i}}{2} \sum_{\ell=1}^{n} \hat{S}^{n-\ell}_{\Delta t} T_{\Delta t}\big(\hat{S}_{\Delta t} + \mathrm{I}\big) u^{\ell-1} \Delta \tilde{W}_{\ell-1} \right\|^{2p}_{\mathbb{H}^\gamma} \\ &\leq K(p)\mathbb{E}\left[\Delta t \sum_{\ell=1}^{n} \|\hat{S}^{n-\ell}_{\Delta t} T_{\Delta t}\big(\hat{S}_{\Delta t} + \mathrm{I}\big) u^{\ell-1}\|^2_{\mathbb{H}^\gamma} \|\phi\|^2_{\mathcal{L}^2(\mathbb{L}^2;\mathbb{H}^\gamma)} \right]^p \\ &\leq K(p)\Delta t \mathbb{E} \sum_{\ell=1}^{n} \|\hat{S}^{n-\ell}_{\Delta t} T_{\Delta t}\big(\hat{S}_{\Delta t} + \mathrm{I}\big) u^{\ell-1}\|^{2p}_{\mathbb{H}^\gamma} \|\phi\|^{2p}_{\mathcal{L}^2(\mathbb{L}^2;\mathbb{H}^\gamma)} \\ &\leq K\Delta t \sum_{\ell=1}^{n} \mathbb{E}\|u^{\ell-1}\|^{2p}_{\mathbb{H}^\gamma}. \end{aligned} \tag{3.6}$$

To estimate the last term on the right-hand side of (3.5), we note that

$$(u^\ell-\hat{S}_{\Delta t}u^{\ell-1})\Delta\tilde{W}_{\ell-1}=-\,\mathrm{i}\Delta tT_{\Delta t}Qu^{\ell-\frac{1}{2}}\Delta\tilde{W}_{\ell-1}-\frac{\mathrm{i}}{2}T_{\Delta t}(\hat{S}_{\Delta t}+\mathrm{I})u^{\ell-1}(\Delta\tilde{W}_{\ell-1})^2$$
$$-\frac{\mathrm{i}}{2}T_{\Delta t}\Big((u^\ell-\hat{S}_{\Delta t}u^{\ell-1})\Delta\tilde{W}_{\ell-1}\Big)\Delta\tilde{W}_{\ell-1}. \tag{3.7}$$

Taking $L^{2p}(\Omega;\mathbb{H}^\gamma)$-norm to obtain

$$\mathbb{E}\big\|(u^\ell-\hat{S}_{\Delta t}u^{\ell-1})\Delta\tilde{W}_{\ell-1}\big\|^{2p}_{\mathbb{H}^\gamma}\le K\Delta t^{2p}(\Delta t\kappa^2)^p\mathbb{E}\|u^{\ell-\frac{1}{2}}\|^{2p}_{\mathbb{H}^\gamma}+K\Delta t^{2p}\mathbb{E}\|u^{\ell-1}\|^{2p}_{\mathbb{H}^\gamma}$$
$$+K(\Delta t\kappa^2)^p\mathbb{E}\|(u^\ell-\hat{S}_{\Delta t}u^{\ell-1})\Delta\tilde{W}_{\ell-1}\|^{2p}_{\mathbb{H}^\gamma}, \tag{3.8}$$

where we use the embedding $\mathbb{H}^1\hookrightarrow\mathbb{L}^\infty$ for $\gamma=0$ or use $\|fg\|_{\mathbb{H}^\gamma}\le K\|f\|_{\mathbb{H}^\gamma}\|g\|_{\mathbb{H}^\gamma}$ for $\gamma\ge 1$. Note that there exists a constant $\Delta t^*>0$ such that $K(\Delta t\kappa^2)^p\le\frac{1}{2}<1$ for $\Delta t\le\Delta t^*$ (here K is the same as the last term on the right-hand side of (3.8)), which leads to

$$\frac{1}{2}\mathbb{E}\big\|(u^\ell-\hat{S}_{\Delta t}u^{\ell-1})\Delta\tilde{W}_{\ell-1}\big\|^{2p}_{\mathbb{H}^\gamma}\le K\Delta t^2\Big(\mathbb{E}\|u^\ell\|^{2p}_{\mathbb{H}^\gamma}+\mathbb{E}\|u^{\ell-1}\|^{2p}_{\mathbb{H}^\gamma}\Big). \tag{3.9}$$

Combining inequalities (3.5), (3.6) and (3.9) together, we have

$$\mathbb{E}\|u^n\|^{2p}_{\mathbb{H}^\gamma}\le K+K\Delta t\sum_{\ell=0}^{n}\mathbb{E}\|u^\ell\|^{2p}_{\mathbb{H}^\gamma},$$

where the positive constant K depends on p, T, operators $\hat{S}_{\Delta t}$ and $T_{\Delta t}$, $\|u^0\|_{H^\gamma}$, ϕ, but does not depend on Δt. The discrete Gronwall's lemma leads to the assertion. □

LEMMA 3.2 Given $\gamma=1,2,\ldots$ and assume $Q\in\mathbb{H}^\gamma$, $u^0\in L^{2p}(\Omega;\mathbb{H}^\gamma)$ and $\phi\in\mathcal{L}_2(\mathbb{L}^2;\mathbb{H}^\gamma)$, then we have holder continuity in temporal direction, i.e., for $p\ge 1$,

$$\mathbb{E}\|u^{n+1}-u^n\|^{2p}_{\mathbb{H}^{\gamma-1}}\le K\Delta t^p,\ \ n=1,2,\ldots,N.$$

Proof. The estimation is similar as the proof of the last term on the right-hand side of (3.5); see estimations (3.7)–(3.9). Start from equation (3.3),

$$u^{n+1}-u^n=(\hat{S}_{\Delta t}-\mathrm{I})u^n-\mathrm{i}\Delta tT_{\Delta t}Qu^{n+\frac{1}{2}}-\mathrm{i}T_{\Delta t}u^{n+\frac{1}{2}}\Delta\tilde{W}_n.$$

Since $\|\hat{S}_{\Delta t}-\mathrm{I}\|_{\mathcal{L}(\mathbb{H}^\gamma,\mathbb{H}^{\gamma-1})}\le K\Delta t^{\frac{1}{2}}$ (see, for instance, De Bouard & Debussche, 2006), we take $L^{2p}(\Omega;\mathbb{H}^{\gamma-1})$-norm on both sides of the above equation and get

$$\mathbb{E}\|u^{n+1}-u^n\|^{2p}_{\mathbb{H}^{\gamma-1}}\le K\Delta t^p\mathbb{E}\|u^n\|^{2p}_{\mathbb{H}^\gamma}+K\Delta t^{2p}\mathbb{E}\Big(\|u^n\|^{2p}_{\mathbb{H}^{\gamma-1}}+\|u^{n+1}\|^{2p}_{\mathbb{H}^{\gamma-1}}\Big)$$
$$+K\Delta t^p\mathbb{E}\|u^n\|^{2p}_{\mathbb{H}^{\gamma-1}}+K(\Delta t\kappa^2)^p\mathbb{E}\|u^{n+1}-u^n\|^{2p}_{\mathbb{H}^{\gamma-1}}, \tag{3.10}$$

there exists a constant $\Delta t^* > 0$ such that $K(\Delta t\kappa^2)^p \le \frac{1}{2} < 1$ for $\Delta t \le \Delta t^*$ (here K is the same as the last term on the right-hand side of (3.10)), which leads to

$$\frac{1}{2}\mathbb{E}\|u^{n+1} - u^n\|^{2p}_{\mathbb{H}^{\gamma-1}} \le K\Delta t^p \mathbb{E}\|u^n\|^{2p}_{\mathbb{H}^{\gamma}} \le K\Delta t^p.$$

This completes the proof. □

Now we are in a position to establish an error estimate of the semidiscrete method (2.6) by virtue of these two lemmas.

THEOREM 3.3 Assume that $u_0 \in L^2(\Omega;\mathbb{H}^3)$, $Q \in \mathbb{H}^3$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2;\mathbb{H}^3)$, then it is of the mean-square order 1, i.e.,

$$\left(\mathbb{E}\|u(t_n) - u^n\|^2_{\mathbb{L}^2}\right)^{1/2} \le K\Delta t.$$

Proof. From (3.4) and (1.1), it follows

$$u^{n+1} = \hat{S}^{n+1}_{\Delta t}u^0 - \mathrm{i}\Delta t\sum_{\ell=1}^{n+1}\hat{S}^{n+1-\ell}_{\Delta t}T_{\Delta t}Qu^{\ell-\frac{1}{2}} - \mathrm{i}\sum_{\ell=1}^{n+1}\hat{S}^{n+1-\ell}_{\Delta t}T_{\Delta t}u^{\ell-\frac{1}{2}}\Delta\tilde{W}_{\ell-1}, \tag{3.11}$$

and

$$\begin{aligned} u(t_{n+1}) &= S(t_{n+1})u^0 - \mathrm{i}\int_0^{t_{n+1}} S(t_{n+1}-\tau)Qu(\tau)\,\mathrm{d}\tau - \mathrm{i}\int_0^{t_{n+1}} S(t_{n+1}-\tau)u(\tau)\circ\mathrm{d}W(\tau) \\ &= S(t_{n+1})u^0 - \mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)Qu(\tau)\,\mathrm{d}\tau - \mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)u(\tau)\circ\mathrm{d}W(\tau). \end{aligned} \tag{3.12}$$

Subtracting (3.11) from (3.12) leads to

$$\begin{aligned} u(t_{n+1}) - u^{n+1} = &\underbrace{\left(S(t_{n+1}) - \hat{S}^{n+1}_{\Delta t}\right)u^0} \\ &- \mathrm{i}\sum_{\ell=1}^{n+1}\left(\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)Qu(\tau)\,\mathrm{d}\tau - \Delta t\hat{S}^{n+1-\ell}_{\Delta t}T_{\Delta t}Qu^{\ell-\frac{1}{2}}\right) \\ &- \mathrm{i}\sum_{\ell=1}^{n+1}\left(\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)u(\tau)\circ\mathrm{d}W(\tau) - \hat{S}^{n+1-\ell}_{\Delta t}T_{\Delta t}u^{\ell-\frac{1}{2}}\Delta\tilde{W}_{\ell-1}\right) \\ =:&\ \mathcal{A} + \mathcal{B} + \mathcal{C}. \end{aligned}$$

We will estimate $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ separately.

• The first term $\mathcal{A}$.
From De Bouard & Debussche (2006), we know that $\|S(t_{n+1}) - \hat{S}_{\Delta t}^{n+1}\|_{\mathcal{L}(\mathbb{H}^3,\mathbb{L}^2)} \leq K\Delta t$. Thus,

$$\mathbb{E}\|\mathcal{A}\|_{\mathbb{L}^2}^2 \leq K\mathbb{E}\|u^0\|_{\mathbb{H}^3}^2 \Delta t^2 \leq K\Delta t^2.$$

• The second term $\mathcal{B}$.
To estimate $\mathcal{B}$, we insert one term

$$\pm\,\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-r)Qu_{t_{\ell-1},u^{\ell-1}}(\tau)\,\mathrm{d}\tau$$

into the expression of $\mathcal{B}$, and we have

$$\begin{aligned}\mathcal{B} = &-\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)Q\Big(u(\tau) - u_{t_{\ell-1},u^{\ell-1}}(\tau)\Big)\,\mathrm{d}\tau\\ &-\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\Big(S(t_{n+1}-r)Qu_{t_{\ell-1},u^{\ell-1}}(\tau) - \hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Qu^{\ell-\frac{1}{2}}\Big)\,\mathrm{d}\tau\\ =:&\ \mathcal{B}^1 + \mathcal{B}^2.\end{aligned}$$

By using the expression of $u(\tau) - u_{t_{\ell-1},u^{\ell-1}}(\tau)$, that is,

$$\begin{aligned}u(\tau) - u_{t_{\ell-1},u^{\ell-1}}(\tau) = &\ S(\tau - t_{\ell-1})(u(t_{\ell-1}) - u^{\ell-1})\\ &-\mathrm{i}\int_{t_{\ell-1}}^{\tau} S(\tau - t_{\ell-1}-\rho)Q\big(u(\rho) - u_{t_{\ell-1},u^{\ell-1}}(\rho)\big)\,\mathrm{d}\rho\\ &-\mathrm{i}\int_{t_{\ell-1}}^{\tau} S(\tau - t_{\ell-1}-\rho)\big(u(\rho) - u_{t_{\ell-1},u^{\ell-1}}(\rho)\big)\circ\mathrm{d}W(\rho),\end{aligned}$$

we have

$$\begin{aligned}\mathbb{E}\|u(\tau) - u_{t_{\ell-1},u^{\ell-1}}(\tau)\|_{\mathbb{L}^2}^2 \leq &\ K\mathbb{E}\|u(t_{\ell-1}) - u^{\ell-1}\|_{\mathbb{L}^2}^2\\ &+K\int_{t_{\ell-1}}^{\tau}\mathbb{E}\|u(\rho) - u_{t_{\ell-1},u^{\ell-1}}(\rho)\|_{\mathbb{L}^2}^2\,\mathrm{d}\rho.\end{aligned}$$

Therefore, Gronwall's inequality leads to

$$\mathbb{E}\|u(\tau) - u_{t_{\ell-1},u^{\ell-1}}(\tau)\|_{\mathbb{L}^2}^2 \leq K\mathbb{E}\|u(t_{\ell-1}) - u^{\ell-1}\|_{\mathbb{L}^2}^2, \tag{3.13}$$

thus for term B^1,

$$\begin{aligned}\mathbb{E}\|\mathcal{B}^1\|_{\mathbb{L}^2}^2 &\le K(n+1)\sum_{\ell=1}^{n+1}\mathbb{E}\left\|\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)Q\big(u(\tau)-u_{t_{\ell-1},u^{\ell-1}}(\tau)\big)\,\mathrm{d}\tau\right\|_{\mathbb{L}^2}^2\\ &\le KT\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\left\|S(t_{n+1}-\tau)Q\big(u(\tau)-u_{t_{\ell-1},u^{\ell-1}}(\tau)\big)\right\|_{\mathbb{L}^2}^2\mathrm{d}\tau\\ &\le K\Delta t\sum_{\ell=1}^{n+1}\mathbb{E}\|u(t_{\ell-1})-u^{\ell-1}\|_{\mathbb{L}^2}^2.\end{aligned}$$

We split term $\mathcal{B}^2$ further as follows

$$\begin{aligned}\mathcal{B}^2 &= -\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\Big(S(t_{n+1}-r)-\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big)Qu_{t_{\ell-1},u^{\ell-1}}(\tau)\,\mathrm{d}\tau\\ &\quad -\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Q\Big(u_{t_{\ell-1},u^{\ell-1}}(\tau)-u^{\ell-1}\Big)\,\mathrm{d}\tau\\ &\quad -\mathrm{i}\Delta t\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Q\Big(u^\ell-u^{\ell-1}\Big)\\ &=: \mathcal{B}_a^2+\mathcal{B}_b^2+\mathcal{B}_c^2.\end{aligned}$$

For term B_a^2, based on $\|S(t_n)-\hat{S}_{\Delta t}^n\|_{\mathcal{L}(\mathbb{H}^3;\mathbb{L}^2)}\le K\Delta t$, $\|\mathrm{I}-T_\Delta\|_{\mathcal{L}(\mathbb{H}^3;\mathbb{L}^2)}\le K\Delta t$ and Lemma 3.1, we have

$$\begin{aligned}\mathbb{E}\|\mathcal{B}_a^2\|_{\mathbb{L}^2}^2 &\le K(n+1)\sum_{\ell=1}^{n+1}\mathbb{E}\left\|\int_{t_{\ell-1}}^{t_\ell}\Big(S(t_{n+1}-r)-\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big)Qu_{t_{\ell-1},u^{\ell-1}}(\tau)\,\mathrm{d}\tau\right\|_{\mathbb{L}^2}^2\\ &\le KT\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\left\|\Big(S(t_{n+1}-r)-\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big)Qu_{t_{\ell-1},u^{\ell-1}}(\tau)\right\|_{\mathbb{L}^2}^2\mathrm{d}\tau\\ &\le KT\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\left\|S(t_{n+1}-r)-\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\right\|_{\mathcal{L}(\mathbb{H}^3;\mathbb{L}^2)}^2\|Q\|_{\mathbb{H}^3}^2\mathbb{E}\|u_{t_{\ell-1},u^{\ell-1}}(\tau)\|_{\mathbb{H}^3}^2\,\mathrm{d}\tau\\ &\le K\Delta t^2.\end{aligned}$$

To estimate term $\mathcal{B}_b^2$, we insert the expression of $u_{t_{\ell-1},u^{\ell-1}}(\tau)-u^{\ell-1}$ into it, and we have

$$\begin{aligned}\mathcal{B}_b^2 = -\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Q\Bigg[&(S(\tau-t_{\ell-1})-\mathrm{I})u^{\ell-1}\\ &-\mathrm{i}\int_{t_{\ell-1}}^{\tau}S(\tau-\rho)\left(Q-\frac{i}{2}\right)u_{t_{\ell-1},u^{\ell-1}}(\rho)\mathrm{d}\rho\Bigg]\,\mathrm{d}\tau\end{aligned}$$

$$
-\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Q\int_{t_{\ell-1}}^{\tau}S(\tau-\rho)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}W(\rho)\,\mathrm{d}\tau.
$$

The estimate of the first term is similar to before and it reads

$$
\begin{aligned}
&\mathbb{E}\Bigg\|-\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Q\Bigg[(S(\tau-t_{\ell-1})-\mathrm{I})u^{\ell-1}\\
&\quad-\mathrm{i}\int_{t_{\ell-1}}^{\tau}S(\tau-\rho)\left(Q-\frac{i}{2}\right)u_{t_{\ell-1},u^{\ell-1}}(\rho)\mathrm{d}\rho\Bigg]\,\mathrm{d}\tau\Bigg\|_{\mathbb{L}^2}^2\\
&\quad\le K\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\left\|(S(\tau-t_{\ell-1})-\mathrm{I})u^{\ell-1}-i\int_{t_{\ell-1}}^{\tau}S(\tau-\rho)\left(Q-\frac{i}{2}\right)u_{t_{\ell-1},u^{\ell-1}}(\rho)\mathrm{d}\rho\right\|_{\mathbb{L}^2}^2\mathrm{d}\tau\\
&\quad\le K\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\left\{\|S(\tau-t_{\ell-1})-\mathrm{I}\|_{\mathcal{L}(\mathbb{H}^2;\mathbb{L}^2)}^2\mathbb{E}\|u^{\ell-1}\|_{\mathbb{H}^2}^2+K\Delta t\int_{t_{\ell-1}}^{\tau}\mathbb{E}\|u_{t_{\ell-1},u^{\ell-1}}(\rho)\|_{\mathbb{L}^2}^2\,\mathrm{d}\rho\right\}\mathrm{d}\tau\\
&\quad\le K\Delta t^2,
\end{aligned}
$$

where in the last step, we use Lemma 3.1 and $\|S(\tau-t_{\ell-1})-\mathrm{I}\|_{\mathcal{L}(\mathbb{H}^2;\mathbb{L}^2)}\le K\Delta t$ (see De Bouard & Debussche, 2006).

Concerning the second term, we employ Fubini's theorem and Itô isometry and Lemma 3.1,

$$
\begin{aligned}
&\mathbb{E}\Bigg\|-\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Q\int_{t_{\ell-1}}^{\tau}S(\tau-\rho)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}W(\rho)\,\mathrm{d}\tau\Bigg\|_{\mathbb{L}^2}^2\\
&\quad=\mathbb{E}\Bigg\|-\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Q\int_{\rho}^{t_\ell}S(\tau-\rho)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}\tau\,\mathrm{d}W(\rho)\Bigg\|_{\mathbb{L}^2}^2\\
&\quad\le\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\mathbb{E}\Bigg\|\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}Q\int_{\rho}^{t_\ell}S(\tau-\rho)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}\tau\Bigg\|_{\mathbb{L}^2}^2\mathrm{d}\rho\\
&\quad\le K\Delta t\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\int_{\rho}^{t_\ell}\mathbb{E}\|u_{t_{\ell-1},u^{\ell-1}}(\rho)\|_{\mathbb{L}^2}^2\,\mathrm{d}\tau\,\mathrm{d}\rho\\
&\quad\le K\Delta t^2.
\end{aligned}
$$

The estimate of term $\mathcal{B}_c^2$ is similar to that of term $\mathcal{B}_b^2$ by replacing the expression of $u^\ell-u^{\ell-1}$. Combining all the above inequalities, we obtain the desired estimate of $\mathcal{B}$

$$
\mathbb{E}\|\mathcal{B}\|_{\mathbb{L}^2}^2\le K\Delta t^2+K\Delta t\sum_{\ell=1}^{n+1}\mathbb{E}\|u(t_{\ell-1})-u^{\ell-1}\|_{\mathbb{L}^2}^2.
$$

- The third term $\mathcal{C}$.

To estimate $\mathcal{C}$, we change Stratonovich integral into Itô one, noting that $F_\phi = \sum_{\ell\in\mathbb{N}^d}\big(\phi e_\ell(x)\big)^2$,

$$\mathcal{C} = -\frac{1}{2}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)u(\tau)F_\phi\,\mathrm{d}\tau - \mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)u(\tau)\,\mathrm{d}W(\tau)$$
$$+\,\mathrm{i}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}u^{\ell-\frac{1}{2}}\,\Delta\tilde{W}_{\ell-1}.$$

We split it further

$$\mathcal{C} = -\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)\Big(u(\tau)-u_{t_{\ell-1},u^{\ell-1}}(\tau)\Big)\,\mathrm{d}W(\tau)$$
$$-\,\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\Big(S(t_{n+1}-\tau)-\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big)u_{t_{\ell-1},u^{\ell-1}}(\tau)\,\mathrm{d}W(\tau)$$
$$-\,\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big(u_{t_{\ell-1},u^{\ell-1}}(\tau)-u^{\ell-1}\Big)\,\mathrm{d}W(\tau) + \frac{\mathrm{i}}{2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big(u^{\ell}-u^{\ell-1}\Big)\Delta\tilde{W}_{\ell-1}$$
$$-\,\frac{1}{2}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)u(\tau)F_\phi\,\mathrm{d}\tau + \mathrm{i}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}u^{\ell-\frac{1}{2}}\Big(\Delta\tilde{W}_{\ell-1}-\Delta W_{\ell-1}\Big).$$

By replacing the expressions of $u_{t_{\ell-1},u^{\ell-1}}(\tau)-u^{\ell-1}$ and $u^{\ell}-u^{\ell-1}$ into the above equation, we have

$$\mathcal{C} = -\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)\Big(u(\tau)-u_{t_{\ell-1},u^{\ell-1}}(\tau)\Big)\,\mathrm{d}W(\tau)$$
$$-\,\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\Big(S(t_{n+1}-\tau)-\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big)u_{t_{\ell-1},u^{\ell-1}}(\tau)\,\mathrm{d}W(\tau)$$
$$-\,\mathrm{i}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Bigg((S(\tau-t_{\ell-1})-\mathrm{I})u^{\ell-1}$$
$$-\mathrm{i}\int_{t_{\ell-1}}^{\tau} S(\tau-\rho)\left(Q-\frac{i}{2}\right)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}\rho\Bigg)\,\mathrm{d}W(\tau)$$
$$+\,\frac{\mathrm{i}}{2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big((\hat{S}_{\Delta t}-\mathrm{I})u^{\ell-1}-i\Delta t T_{\Delta t}Qu^{\ell-\frac{1}{2}}\Big)\Delta\tilde{W}_{\ell-1}$$
$$-\,\frac{1}{2}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau)\Big(u(\tau)-u_{t_{\ell-1},u^{\ell-1}}(\rho)\Big)F_\phi\,\mathrm{d}\tau$$
$$-\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\int_{t_{\ell-1}}^{\tau} S(\tau-\rho)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}W(\rho)\,\mathrm{d}W(\tau)$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}^{2}u^{\ell-\frac{1}{2}}(\Delta\tilde{W}_{\ell-1})^{2}\\
&-\frac{1}{2}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}S(t_{n+1}-\tau)u_{t_{\ell-1},u^{\ell-1}}(\rho)F_{\phi}\,\mathrm{d}\tau\\
&+\mathrm{i}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}u^{\ell-\frac{1}{2}}\Big(\Delta\tilde{W}_{\ell-1}-\Delta W_{\ell-1}\Big)\\
=\;&\mathcal{C}^1+\mathcal{C}^2+\mathcal{C}^3+\mathcal{C}^4+\mathcal{C}^5,
\end{aligned}
$$

where $\mathcal{C}^j$ denotes terms in the jth lines for $j=1,\ldots,5$.

The estimates of $\mathcal{C}^1$, $\mathcal{C}^2$ and $\mathcal{C}^3$ are similar as before. Take $\mathcal{C}^1$ as an example, via Itô isometry, we have

$$
\begin{aligned}
\mathbb{E}\Big\|\mathcal{C}^1\Big\|_{\mathbb{L}^2}^2&\le K\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\Big[\mathbb{E}\|u(\tau)-u_{t_{\ell-1},u^{\ell-1}}(\tau)\|_{\mathbb{L}^2}^2\\
&\qquad+\mathbb{E}\Big\|\Big(S(t_{n+1}-\tau)-\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\Big)u_{t_{\ell-1},u^{\ell-1}}(\tau)\Big\|_{\mathbb{L}^2}^2\Big]\,\mathrm{d}\tau\\
&\le K\Delta t^2+K\Delta t\sum_{\ell=1}^{n+1}\mathbb{E}\|u(t_{\ell-1})-u^{\ell-1}\|_{\mathbb{L}^2}^2,
\end{aligned}
$$

where in the last step we utilize (3.13), the estimate of operators $\hat{S}_{\Delta t}$, S and $T_{\Delta t}$, and Lemma 3.1. Similarly, we may obtain

$$
\mathbb{E}\Big\|\mathcal{C}^2+\mathcal{C}^3\Big\|_{\mathbb{L}^2}^2\le K\Delta t^2+K\Delta t\sum_{\ell=1}^{n+1}\mathbb{E}\|u(t_{\ell-1})-u^{\ell-1}\|_{\mathbb{L}^2}^2.
$$

We estimate $\mathcal{C}^4$ and $\mathcal{C}^5$ together, since the estimate of them is much technique. First, for the first term in $\mathcal{C}^4$, we have

$$
\begin{aligned}
&-\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\int_{t_{\ell-1}}^{\tau}S(\tau-\rho)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}W(\rho)\,\mathrm{d}W(\tau)\\
=&-\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\int_{t_{\ell-1}}^{\tau}\Big(S(\tau-\rho)-T_{\Delta t}\Big)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}W(\rho)\,\mathrm{d}W(\tau)\\
&-\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\int_{t_{\ell-1}}^{\tau}T_{\Delta t}\Big(u_{t_{\ell-1},u^{\ell-1}}(\rho)-u^{\ell-1}\Big)\,\mathrm{d}W(\rho)\,\mathrm{d}W(\tau)\\
&-\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}^2u^{\ell-1}\int_{t_{\ell-1}}^{t_\ell}\int_{t_{\ell-1}}^{\tau}\mathrm{d}W(\rho)\,\mathrm{d}W(\tau).
\end{aligned}
$$

We claim that the last term in the above equality has the form

$$\sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} \mathrm{d}W(\rho)\,\mathrm{d}W(\tau)$$
$$= \frac{1}{2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \Big((\Delta W_{\ell-1})^2 - F_\phi \Delta t \Big).$$

In fact,

$$\begin{aligned}
&\sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} \mathrm{d}W(\rho)\,\mathrm{d}W(\tau)\\
&= \sum_{k_1,k_2\in\mathbb{N}} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} \mathrm{d}\beta_{k_1}(\rho)\,\mathrm{d}\beta_{k_2}(\tau)\\
&= \sum_{k_1=k_2\in\mathbb{N}} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} \mathrm{d}\beta_{k_1}(\rho)\,\mathrm{d}\beta_{k_1}(\tau)\\
&\quad+ \sum_{k_1<k_2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} \mathrm{d}\beta_{k_1}(\rho)\,\mathrm{d}\beta_{k_2}(\tau)\\
&\quad+ \sum_{k_1>k_2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} \mathrm{d}\beta_{k_1}(\rho)\,\mathrm{d}\beta_{k_2}(\tau)\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{3.14}$$

Because of

$$\begin{aligned}
\int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} \mathrm{d}\beta(\rho)\,\mathrm{d}\beta(\tau) &= \int_{t_{\ell-1}}^{t_\ell} \beta(\tau)\mathrm{d}\beta(\tau) - \beta(t_{\ell-1})(\Delta\beta)\\
&= \frac{1}{2}\Big(\beta^2(t_\ell) - \beta^2(t_{\ell-1})\Big) - \frac{1}{2}\Delta t - \beta(t_{\ell-1})(\Delta\beta) = \frac{1}{2}\Big((\Delta\beta)^2 - \Delta t\Big),
\end{aligned}$$

with $\beta(t)$ being a standard Brownian motion and $\Delta\beta = \beta(t_\ell) - \beta(t_{\ell-1})$, we have

$$\mathrm{I} = \frac{1}{2} \sum_{k_1=k_2\in\mathbb{N}} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \Big((\Delta\beta_{k_1})^2 - \Delta t \Big).$$

We change the index of k_1 and k_2 in the last term of (3.14) to obtain

$$\mathrm{III} = \sum_{k_2>k_1} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_2} \phi e_{k_1} \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} \mathrm{d}\beta_{k_2}(\rho)\,\mathrm{d}\beta_{k_1}(\tau)$$

and

$$\begin{aligned}\mathrm{II}+\mathrm{III} &= \sum_{k_1<k_2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}^2 u^{\ell-1}\phi e_{k_1}\phi e_{k_2} \\ &\quad\times\left[\int_{t_{\ell-1}}^{t_\ell}\int_{t_{\ell-1}}^{\tau}\mathrm{d}\beta_{k_1}(\rho)\,\mathrm{d}\beta_{k_2}(\tau)+\int_{t_{\ell-1}}^{t_\ell}\int_{t_{\ell-1}}^{\tau}\mathrm{d}\beta_{k_2}(\rho)\,\mathrm{d}\beta_{k_1}(\tau)\right] \\ &= \sum_{k_1<k_2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}^2 u^{\ell-1}\phi e_{k_1}\phi e_{k_2}\Delta\beta_{k_1}\Delta\beta_{k_2}.\end{aligned}$$

Combining them together we may prove the claim.

After the rearrangement of $\mathcal{C}^4+\mathcal{C}^5$, we have

$$\begin{aligned}\mathcal{C}^4+\mathcal{C}^5 &= -\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\int_{t_{\ell-1}}^{\tau}\Big(S(\tau-\rho)-T_{\Delta t}\Big)u_{t_{\ell-1},u^{\ell-1}}(\rho)\,\mathrm{d}W(\rho)\,\mathrm{d}W(\tau) \\ &\quad-\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}\int_{t_{\ell-1}}^{\tau}T_{\Delta t}\Big(u_{t_{\ell-1},u^{\ell-1}}(\rho)-u^{\ell-1}\Big)\,\mathrm{d}W(\rho)\,\mathrm{d}W(\tau) \\ &\quad-\frac{1}{2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}^2u^{\ell-1}\big((\Delta W_{\ell-1})^2-(\Delta\hat{W}_{\ell-1})^2\big) \\ &\quad+\frac{1}{4}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}^2(u^\ell-u^{\ell-1})(\Delta\hat{W}_{\ell-1})^2 \\ &\quad-\frac{1}{2}\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}S(t_{n+1}-\tau)u_{t_{\ell-1},u^{\ell-1}}(\rho)F_\phi\,\mathrm{d}\tau+\frac{1}{2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}^2u^{\ell-1}F_\phi\Delta t \\ &\quad+i\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}u^{\ell-\frac{1}{2}}\Big(\Delta\tilde{W}_{\ell-1}-\Delta W_{\ell-1}\Big) \\ &= \mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3+\mathcal{D}_4+\mathcal{D}_5,\end{aligned}$$

where $\mathcal{D}_j$ denotes terms in jth line for $j=1,\ldots,5$. The estimates of $\mathcal{D}_1+\mathcal{D}_2$ come from Itô isometry and Lemma 3.1, that is,

$$\begin{aligned}\mathbb{E}\|\mathcal{D}_1+\mathcal{D}_2\|_{\mathbb{L}^2}^2 &\le K\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\int_{t_{\ell-1}}^{\tau}\|S(\tau-\rho)-T_{\Delta t}\|_{\mathcal{L}(\mathbb{H}^1;\mathbb{L}^2)}^2\|u_{t_{\ell-1},u^{\ell-1}}(\rho)\|_{\mathbb{H}^1}^2\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &\quad+K\sum_{\ell=1}^{n+1}\int_{t_{\ell-1}}^{t_\ell}\int_{t_{\ell-1}}^{\tau}+\|u_{t_{\ell-1},u^{\ell-1}}(\rho)-u^{\ell-1}\|_{\mathbb{L}^2}^2\,\mathrm{d}\rho\,\mathrm{d}\tau \\ &\le K\Delta t^2.\end{aligned}$$

The estimate of the first term in $\mathcal{D}_3$ is benefit from properties (2.2) of truncated Wiener process,

$$\begin{aligned}
&\mathbb{E}\left\| -\frac{1}{2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}^2u^{\ell-1}\big((\Delta W_{\ell-1})^2-(\Delta\hat{W}_{\ell-1})^2\big)\right\|_{\mathbb{L}^2}^2\\
&\le K\mathbb{E}\sum_{\ell_1=1}^{n+1}\sum_{\ell_2=1}^{n+1}\|u^{\ell_1-1}\|_{\mathbb{L}^2}\|u^{\ell_2-1}\|_{\mathbb{L}^2}\|(\Delta W_{\ell_1-1})^2-(\Delta\hat{W}_{\ell_1-1})^2\|_{\mathbb{H}^1}\|(\Delta W_{\ell_2-1})^2-(\Delta\hat{W}_{\ell_2-1})^2\|_{\mathbb{H}^1}\\
&\le K\Delta t^2.
\end{aligned}$$

By inserting the expression of $u^\ell - u^{\ell-1}$ into the second term in $\mathcal{D}_3$ and estimating as before, we could get $\mathbb{E}\|\mathcal{D}_3\|_{\mathbb{L}^2}^2 \le K\Delta t^2$. The estimate of $\mathcal{D}_3$ is similar to that of $\mathcal{B}^2$ and is bounded also by $K\Delta t^2$. For the term $\mathcal{D}_5$, we split it further to get

$$\begin{aligned}
\mathcal{D}_5 = \mathrm{i}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}u^{\ell-1}\Big(\Delta\tilde{W}_{\ell-1}-\Delta W_{\ell-1}\Big)\\
+\frac{\mathrm{i}}{2}\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}(u^\ell-u^{\ell-1})\Big(\Delta\tilde{W}_{\ell-1}-\Delta W_{\ell-1}\Big).
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
&\mathbb{E}\|i\sum_{\ell=1}^{n+1}\hat{S}_{\Delta t}^{n+1-\ell}T_{\Delta t}u^{\ell-1}\Big(\Delta\tilde{W}_{\ell-1}-\Delta W_{\ell-1}\Big)\|_{\mathbb{L}^2}^2\\
&\le K\sum_{\ell=1}^{n+1}\mathbb{E}(\|u^{\ell-1}\|_{\mathbb{L}^2}^2)\mathbb{E}(\|\Delta\tilde{W}_{\ell-1}-\Delta W_{\ell-1}\|_{\mathbb{H}^1}^2)\le K\Delta t^2.
\end{aligned}$$

The estimate of the second term follows from inserting the expression of $u^\ell - u^{\ell-1}$ and estimating similarly, finally, we have $\mathbb{E}\|\mathcal{D}_5\|_{\mathbb{H}^1}^2 \le K\Delta t^2$.

Combining all these analysis above, we obtain

$$\mathbb{E}\|u(t_{n+1})-u^{n+1}\|_{\mathbb{L}^2}^2 \le K\Delta t^2 + K\Delta t\sum_{\ell=1}^{n+1}\|u(t_{\ell-1})-u^{\ell-1}\|_{\mathbb{L}^2}^2.$$

Therefore, Gronwall's lemma leads to the assertion. □

3.2 *Spatial error*

We state the spatial error estimate of the symplectic local discontinuous Galerkin method (2.7) for the stochastic linear Schrödinger equation (1.1).

THEOREM 3.4 Assume $u_0 \in L^2(\Omega; \mathbb{H}^{k+2})$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2; \mathbb{H}^{k+2})$. Let u_h^n be the numerical solution of the symplectic local discontinuous Galerkin method (2.7). Then there exists a constant $h_0 > 0$ such that for $h \le h_0$,

$$\mathbb{E}\|u^n - u_h^n\|_{\mathbb{L}^2}^2 \le Ch^{2k+2} + C\Delta t^{-1} h^{2k+2}. \tag{3.15}$$

Proof. Notice that the method (2.7) is also satisfied when the numerical solutions r_h, p_h, s_h, q_h are replaced by the exact solutions $r, p = s_x, s, q = s_x$. For each fixed t_n, we can obtain the cell error equation

$$\begin{aligned}
&\mathfrak{B}_j(r^n - r_h^n, p^n - p_h^n, s^n - s_h^n, q^n - q_h^n; \nu_h, \omega_h, \alpha_h, \beta_h)\\
&= \int_{I_j} [r^{n+1} - r_h^{n+1}]\nu_h \,\mathrm{d}x - \int_{I_j} [r^n - r_h^n]\nu_h \,\mathrm{d}x + \Delta t \int_{I_j} (p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}})(\nu_h)_x \,\mathrm{d}x\\
&- \int_{I_j} (s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}})\nu_h \Delta\tilde{W}_n \,\mathrm{d}x - \Delta t \int_{I_j} (p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}})\omega_h \,\mathrm{d}x\\
&- \Delta t \int_{I_j} (s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}})(\omega_h)_x \,\mathrm{d}x\\
&- \Delta t \int_{I_j} (s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}})Q_h\nu_h \,\mathrm{d}x + \Delta t \int_{I_j} (r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}})Q_h\alpha_h \,\mathrm{d}x\\
&- \Delta t \int_{I_j} (q^{n+\frac{1}{2}} - q_h^{n+\frac{1}{2}})(\alpha_h)_x \,\mathrm{d}x + \int_{I_j} [s^{n+1} - s_h^{n+1}]\alpha_h \,\mathrm{d}x - \int_{I_j} [s^n - s_h^n]\alpha_h \,\mathrm{d}x\\
&+ \int_{I_j} (r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}})\alpha_h \Delta\tilde{W}_n \,\mathrm{d}x - \Delta t \int_{I_j} (q^{n+\frac{1}{2}} - q_h^{n+\frac{1}{2}})\beta_h \,\mathrm{d}x\\
&- \Delta t \int_{I_j} (r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}})(\beta_h)_x \,\mathrm{d}x\\
&- \Delta t[(p^{n+\frac{1}{2}} - \hat{p}^{n+\frac{1}{2}})\nu_h^-]_{j+\frac{1}{2}} + \Delta t[(p^{n+\frac{1}{2}} - \hat{p}^{n+\frac{1}{2}})\nu_h^+]_{j-\frac{1}{2}} + \Delta t[(s^{n+\frac{1}{2}} - \hat{s}^{n+\frac{1}{2}})\omega_h^-]_{j+\frac{1}{2}}\\
&- \Delta t[(s^{n+\frac{1}{2}} - \hat{s}^{n+\frac{1}{2}})\omega_h^+]_{j-\frac{1}{2}} + \Delta t[(q^{n+\frac{1}{2}} - \hat{q}^{n+\frac{1}{2}})\alpha_h^-]_{j+\frac{1}{2}} - \Delta t[(q^{n+\frac{1}{2}} - \hat{q}^{n+\frac{1}{2}})\alpha_h^+]_{j-\frac{1}{2}}\\
&+ \Delta t[(r^{n+\frac{1}{2}} - \hat{r}^{n+\frac{1}{2}})\beta_h^-]_{j+\frac{1}{2}} - \Delta t[(r^{n+\frac{1}{2}} - \hat{r}^{n+\frac{1}{2}})\beta_h^+]_{j-\frac{1}{2}} = 0
\end{aligned} \tag{3.16}$$

for all $\nu_h, \omega_h, \alpha_h, \beta_h \in V_h^k$.

Summing over j, the error equation becomes

$$\sum_{j=1}^{J} \mathfrak{B}_j(r^n - r_h^n, p^n - p_h^n, s^n - s_h^n, q^n - q_h^n; \nu_h, \omega_h, \alpha_h, \beta_h) = 0 \tag{3.17}$$

for all $\nu_h, \omega_h, \alpha_h, \beta_h \in V_h^k$.

Denoting

$$\varepsilon^n = \mathcal{P}^- r^n - r_h^n,\ \xi^n = \mathcal{P} q^n - q_h^n,\ \eta^n = \mathcal{P}^- s^n - s_h^n,\ \zeta^n = p_h^n - \mathcal{P} p^n,$$

$$\varepsilon_e^n = \mathcal{P}^- r^n - r^n,\ \xi_e^n = \mathcal{P} q^n - q^n,\ \eta_e^n = \mathcal{P}^- s^n - s^n,\ \zeta_e^n = p^n - \mathcal{P} p^n, \tag{3.18}$$

where $\mathcal{P}$ is the standard $\mathbb{L}^2$-projection of a function ω with $k+1$ continuous derivatives into space V_h^k, $\mathcal{P}^-$ is a special projector into V_h^k, which satisfies, for each j,

$$\int_{\mathrm{I}_j} (\mathcal{P}^- \omega(x) - \omega(x)) \nu(x)\, \mathrm{d}x = 0 \ \ \forall \nu \in P^{k-1}(\mathrm{I}_j),$$

and $\mathcal{P}^-(\omega(x^-_{j+\frac{1}{2}})) = \omega(x_{j+\frac{1}{2}})$ and taking the test functions

$$\nu_h = \varepsilon^{n+\frac{1}{2}},\ \omega_h = \xi^{n+\frac{1}{2}},\ \alpha_h = \eta^{n+\frac{1}{2}},\ \beta_h = \zeta^{n+\frac{1}{2}},$$

we obtain the important energy equality

$$\sum_{j=1}^{J} \mathfrak{B}_j(\varepsilon^n - \varepsilon_e^n, \zeta_e^n - \zeta^n, \eta^n - \eta_e^n, \xi^n - \xi_e^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) = 0. \tag{3.19}$$

Now, we shall prove the theorem by analysing each terms of (3.19).

We consider the left-hand side of the energy equation (3.19). Using the linearity of $\mathfrak{B}_j$ with respect to its first group of arguments, we get

$$\begin{aligned} &\mathfrak{B}_j(\varepsilon^n - \varepsilon_e^n, \zeta_e^n - \zeta^n, \eta^n - \eta_e^n, \xi^n - \xi_e^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) \\ &\quad = \mathfrak{B}_j(\varepsilon^n, -\zeta^n, \eta^n, \xi^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) \\ &\qquad - \mathfrak{B}_j(\varepsilon_e^n, -\zeta_e^n, \eta_e^n, \xi_e^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}). \end{aligned} \tag{3.20}$$

First, we consider the first term of the right-hand side in (3.20), which yields

$$\begin{aligned} &\mathfrak{B}_j(\varepsilon^n, -\zeta^n, \eta^n, \xi^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) \\ &\quad = \frac{1}{2}\int_{\mathrm{I}_j} \left((\varepsilon^{n+1})^2 - (\varepsilon^n)^2\right) \mathrm{d}x + \frac{1}{2}\int_{\mathrm{I}_j} \left((\eta^{n+1})^2 - (\eta^n)^2\right) \mathrm{d}x \\ &\qquad + \Delta t\Big[(\zeta^+\varepsilon^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\zeta^+\varepsilon^+)^{n+\frac{1}{2}}_{j-\frac{1}{2}}\Big] + \Delta t\Big[(\eta^-\xi^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\eta^-\xi^+)^{n+\frac{1}{2}}_{j-\frac{1}{2}}\Big] \\ &\qquad + \Delta t\Big[(\xi^+\eta^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\xi^+\eta^+)^{n+\frac{1}{2}}_{j-\frac{1}{2}}\Big] + \Delta t\Big[(\varepsilon^-\zeta^-)^{n+\frac{1}{2}}_{j+\frac{1}{2}} - (\varepsilon^-\zeta^+)^{n+\frac{1}{2}}_{j-\frac{1}{2}}\Big] \\ &\qquad - \Delta t \underbrace{\int_{\mathrm{I}_j} [(\eta\xi)_x^{n+\frac{1}{2}} + (\varepsilon\zeta)_x^{n+\frac{1}{2}}]\, \mathrm{d}x}_{R}. \end{aligned} \tag{3.21}$$

Applying integration by parts, we arrive at

$$R=\Big[(\eta^-\xi^-)^{n+\frac12}_{j+\frac12}-(\eta^+\xi^+)^{n+\frac12}_{j-\frac12}\Big]+\Big[(\varepsilon^-\zeta^-)^{n+\frac12}_{j+\frac12}-(\varepsilon^+\zeta^+)^{n+\frac12}_{j-\frac12}\Big]. \tag{3.22}$$

Substituting (3.22) into (3.21), we have

$$\begin{aligned}&\mathfrak{B}_j(\varepsilon^n,-\zeta^n,\eta^n,\xi^n;\varepsilon^{n+\frac12},\xi^{n+\frac12},\eta^{n+\frac12},\zeta^{n+\frac12})\\ &=\frac12\int_{\mathrm{I}_j}\Big((\varepsilon^{n+1})^2-(\varepsilon^n)^2\Big)\,\mathrm{d}x+\frac12\int_{\mathrm{I}_j}\Big((\eta^{n+1})^2-(\eta^n)^2\Big)\,\mathrm{d}x+\Delta t[\hat{\Phi}^{n+\frac12}_{j+\frac12}-\hat{\Phi}^{n+\frac12}_{j-\frac12}],\end{aligned} \tag{3.23}$$

where

$$\hat{\Phi}^{n+\frac12}_{j-\frac12}=(\xi^{n+\frac12,+}\eta^{n+\frac12,-})_{j-\frac12}+(\zeta^{n+\frac12,+}\varepsilon^{n+\frac12,-})_{j-\frac12},$$
$$\hat{\Phi}^{n+\frac12}_{j+\frac12}=(\xi^{n+\frac12,+}\eta^{n+\frac12,-})_{j+\frac12}+(\zeta^{n+\frac12,+}\varepsilon^{n+\frac12,-})_{j+\frac12}.$$

As for the second term of the right-hand side in (3.20), we have

$$\mathfrak{B}_j(\varepsilon_e^n,-\zeta_e^n,\eta_e^n,\xi_e^n;\varepsilon^{n+\frac12},\xi^{n+\frac12},\eta^{n+\frac12},\zeta^{n+\frac12})=\mathrm{I}+\mathrm{II}+\mathrm{III}+IV+V, \tag{3.24}$$

where

$$\begin{aligned}\mathrm{I}&=\int_{\mathrm{I}_j}(\varepsilon_e^{n+1}-\varepsilon_e^n)\varepsilon^{n+\frac12}\,\mathrm{d}x+\int_{\mathrm{I}_j}(\eta_e^{n+1}-\eta_e^n)\eta^{n+\frac12}\,\mathrm{d}x,\\ \mathrm{II}&=\Delta t\int_{\mathrm{I}_j}\Big((\zeta_e\xi)^{n+\frac12}-(\zeta_e\varepsilon_x)^{n+\frac12}-(\eta_e\xi_x)^{n+\frac12}-(\xi_e\eta_x)^{n+\frac12}\\ &\qquad-(\xi_e\zeta)^{n+\frac12}-(\varepsilon_e\zeta_x)^{n+\frac12}-Q_h(\eta_e\varepsilon)^{n+\frac12}+Q_h(\varepsilon_e\eta)^{n+\frac12}\Big)\,\mathrm{d}x,\\ \mathrm{III}&=\int_{\mathrm{I}_j}\varepsilon_e^{n+\frac12}\eta^{n+\frac12}\Delta\tilde{W}_n\,\mathrm{d}x,\quad \mathrm{IV}=-\int_{\mathrm{I}_j}\eta_e^{n+\frac12}\varepsilon^{n+\frac12}\Delta\tilde{W}_n\,\mathrm{d}x,\\ V&=\Delta t\Big[(\zeta_e^+\varepsilon^-)^{n+\frac12}_{j+\frac12}-(\zeta_e^+\varepsilon^+)^{n+\frac12}_{j-\frac12}-(\eta_e^-\xi^-)^{n+\frac12}_{j+\frac12}+(\eta_e^-\xi^+)^{n+\frac12}_{j-\frac12}\\ &\qquad-(\xi_e^e\eta^-)^{n+\frac12}_{j+\frac12}-(\xi_e^+\eta^+)^{n+\frac12}_{j-\frac12}-(\varepsilon_e^-\zeta^-)^{n+\frac12}_{j+\frac12}+(\varepsilon_e^-\zeta^+)^{n+\frac12}_{j-\frac12}\Big].\end{aligned}$$

By using the simple inequality $ab\le\frac{a^2}{4}+b^2$, we have

$$\begin{aligned}\mathrm{I}&\le\|\varepsilon_e^{n+1}-\varepsilon_e^n\|_{\mathbb{L}^2(\mathrm{I}_j)}\|\varepsilon^{n+\frac12}\|_{\mathbb{L}^2(\mathrm{I}_j)}+\|\eta_e^{n+1}-\eta_e^n\|_{\mathbb{L}^2(\mathrm{I}_j)}\|\eta^{n+\frac12}\|_{\mathbb{L}^2(\mathrm{I}_j)}\\ &\le C\Delta t^{-1}\|\varepsilon_e^{n+1}-\varepsilon_e^n\|^2_{\mathbb{L}^2(\mathrm{I}_j)}+C\Delta t\|\varepsilon^{n+\frac12}\|^2_{\mathbb{L}^2(\mathrm{I}_j)}\\ &\quad+C\Delta t^{-1}\|\eta_e^{n+1}-\eta_e^n\|^2_{\mathbb{L}^2(\mathrm{I}_j)}+C\Delta t\|\eta^{n+\frac12}\|^2_{\mathbb{L}^2(\mathrm{I}_j)}.\end{aligned} \tag{3.25}$$

A well-known result of the finite element theory is the following approximation property: for all functions $\omega \in \mathbb{H}^{k+1}(\mathcal{O})$ (see Theorem 3.1.6 in Ciarlet (1978))

$$\|\breve{\omega}(x)\|_{\mathbb{L}^2} + h\|\breve{\omega}(x)\|_{\mathbb{L}^\infty} + \sqrt{h}\|\breve{\omega}(x)\|_{\Gamma_h} \leq C\|\omega\|_{\mathbb{H}^{k+1}} h^{k+1}, \tag{3.26}$$

where $\breve{\omega} = \mathcal{P}\omega - \omega$ or $\breve{\omega} = \mathcal{P}^-\omega - \omega$. The positive constant C is independent of h, and Γ_h is the usual L^2-norm on the cell interfaces of the mesh, which for this one-dimensional case is $\|\nu\|^2_{\Gamma_h} = \sum_{j=1}^{J}\left((\nu^-_{j+\frac{1}{2}})^2 + (\nu^+_{j-\frac{1}{2}})^2\right)$.

Summing over j and taking expectation of (3.25), we get

$$\mathbb{E}\left(\sum_{j=1}^{J}\mathrm{I}\right) \leq C\Delta t^{-1}\mathbb{E}\|\varepsilon_e^{n+1} - \varepsilon_e^n\|^2_{\mathbb{L}^2([L_f,L_r])} + C\Delta t\mathbb{E}\|\varepsilon^{n+\frac{1}{2}}\|^2_{\mathbb{L}^2([L_f,L_r])}$$
$$+ C\Delta t^{-1}\mathbb{E}\|\eta_e^{n+1} - \eta_e^n\|^2_{\mathbb{L}^2([L_f,L_r])} + C\Delta t\mathbb{E}\|\eta^{n+\frac{1}{2}}\|^2_{\mathbb{L}^2([L_f,L_r])}.$$

Recalling that $\varepsilon_e^{n+1} - \varepsilon_e^n = \mathcal{P}^-(r^{n+1} - r^n) - (r^{n+1} - r^n)$ and $\eta_e^{n+1} - \eta_e^n = \mathcal{P}^-(s^{n+1} - s^n) - (s^{n+1} - s^n)$, we replace ω in (3.26) with $r^{n+1} - r^n$ and $s^{n+1} - s^n$, respectively, and use the estimate of $\mathbb{E}\|r^{n+1} - r^n\|^2_{\mathbb{H}^{k+1}([L_f,L_r])}$ and $\mathbb{E}\|s^{n+1} - s^n\|^2_{\mathbb{H}^{k+1}([L_f,L_r])}$ (see Lemma 3.2 with $p = 1$) and Lemma 3.1, we have

$$\mathbb{E}\|\varepsilon_e^{n+1} - \varepsilon_e^n\|^2_{\mathbb{L}^2([L_f,L_r])} + \mathbb{E}\|\eta_e^{n+1} - \eta_e^n\|^2_{\mathbb{L}^2([L_f,L_r])}$$
$$\leq Ch^{2k+2}\left(\mathbb{E}\|r^{n+1} - r^n\|^2_{\mathbb{H}^{k+1}([L_f,L_r])} + \mathbb{E}\|s^{n+1} - s^n\|^2_{\mathbb{H}^{k+1}}\right)$$
$$\leq C\mathbb{E}\|u_0\|^2_{\mathbb{H}^{k+2}([L_f,L_r])} h^{2k+2}\Delta t.$$

Thus for term I, we obtain

$$\mathbb{E}\left(\sum_{j=1}^{J}\mathrm{I}\right) \leq C\mathbb{E}\|u_0\|^2_{\mathbb{H}^{k+2}([L_f,L_r])} h^{2k+2} + C\Delta t\mathbb{E}\|\varepsilon^{n+\frac{1}{2}}\|^2_{\mathbb{L}^2([L_f,L_r])}$$
$$+ C\Delta t\mathbb{E}\|\eta^{n+\frac{1}{2}}\|^2_{\mathbb{L}^2([L_f,L_r])}. \tag{3.27}$$

From the property of the projections $\mathcal{P}$ and $\mathcal{P}^-$, it follows that all the terms in II except the last two terms are actually zero. We can get the estimates for II via Young's inequality and Lemma 3.1,

$$\mathbb{E}\left(\sum_{j=1}^{J}\mathrm{II}\right) \leq C\mathbb{E}\left(\|r^n\|^2_{\mathbb{H}^{k+2}} + \|s^n\|^2_{\mathbb{H}^{k+2}}\right)\Delta t h^{2k+2}$$
$$+ \frac{\Delta t}{4}\mathbb{E}\|\varepsilon^{n+\frac{1}{2}}\|^2_{\mathbb{L}^2([L_f,L_r])} + \frac{\Delta t}{4}\mathbb{E}\|\eta^{n+\frac{1}{2}}\|^2_{\mathbb{L}^2([L_f,L_r])}$$
$$\leq C\mathbb{E}\|u_0\|^2_{\mathbb{H}^{k+2}}\Delta t h^{2k+2} + \frac{\Delta t}{4}\mathbb{E}\|\varepsilon^{n+\frac{1}{2}}\|^2_{\mathbb{L}^2([L_f,L_r])} + \frac{\Delta t}{4}\mathbb{E}\|\eta^{n+\frac{1}{2}}\|^2_{\mathbb{L}^2([L_f,L_r])}.$$

For the third term III, we have

$$\mathbb{E}\left(\sum_{j=1}^{J}\mathrm{III}\right)=\frac{1}{4}\mathbb{E}\int_{L_f}^{L_r}(\varepsilon_e^{n+1}-\varepsilon_e^{n})(\eta^{n+1}-\eta^{n})\Delta\tilde{W}_n\,\mathrm{d}x$$
$$+\frac{1}{2}\mathbb{E}\int_{L_f}^{L_r}(\varepsilon_e^{n+1}-\varepsilon_e^{n})\eta^{n}\Delta\tilde{W}_n\,\mathrm{d}x+\frac{1}{2}\mathbb{E}\int_{L_f}^{L_r}\varepsilon_e^{n}(\eta^{n+1}-\eta^{n})\Delta\tilde{W}_n\,\mathrm{d}x$$
$$=:\mathrm{III}^{\mathrm{a}}+\mathrm{III}^{\mathrm{b}}+\mathrm{III}^{\mathrm{c}}.$$

For term $\mathrm{III}^{\mathrm{a}}$, using Young's inequality, Lemmas 3.1 and 3.2 with $p=2$, we have

$$\begin{aligned}\mathrm{III}^{\mathrm{a}}&\le\frac{1}{4}\mathbb{E}\Big(\|\varepsilon_e^{n+1}-\varepsilon_e^{n}\|_{\mathbb{L}^2([L_f,L_r])}\|\eta^{n+1}-\eta^{n}\|_{\mathbb{L}^2([L_f,L_r])}\|\Delta\tilde{W}_n\|_{\mathbb{L}^\infty([L_f,L_r])}\Big)\\&\le C\Delta t\mathbb{E}\|\eta^{n+1}-\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\Delta t^{-1}\mathbb{E}\Big(\|\Delta\tilde{W}_n\|^2_{\mathbb{L}^\infty([L_f,L_r])}\|\varepsilon_e^{n+1}-\varepsilon_e^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}\Big)\\&\le C\Delta t\mathbb{E}\|\eta^{n+1}-\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}\\&\quad+C\Delta t^{-1}\Big(h^{2k+2}\mathbb{E}\|\Delta\tilde{W}_n\|^4_{\mathbb{L}^\infty([L_f,L_r])}+h^{-(2k+2)}\mathbb{E}\|\varepsilon_e^{n+1}-\varepsilon_e^{n}\|^4_{\mathbb{L}^2([L_f,L_r])}\Big)\\&\le C\Delta t\mathbb{E}\|\eta^{n+1}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\Delta t\mathbb{E}\|\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\Delta th^{2k+2}.\end{aligned}$$

Similarly, for term $\mathrm{III}^{\mathrm{b}}$,

$$\begin{aligned}\mathrm{III}^{\mathrm{b}}&\le\frac{1}{2}\mathbb{E}\Big(\|\varepsilon_e^{n+1}-\varepsilon_e^{n}\|_{\mathbb{L}^2([L_f,L_r])}\|\eta^{n}\|_{\mathbb{L}^2([L_f,L_r])}\|\Delta\tilde{W}_n\|_{\mathbb{L}^\infty([L_f,L_r])}\Big)\\&\le C\mathbb{E}\|\varepsilon_e^{n+1}-\varepsilon_e^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\mathbb{E}\Big(\|\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}\|\Delta\tilde{W}_n\|^2_{\mathbb{L}^\infty([L_f,L_r])}\Big)\\&\le C\Delta t\mathbb{E}\|\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\Delta th^{2k+2}\end{aligned}$$

and for term $\mathrm{III}^{\mathrm{c}}$,

$$\begin{aligned}\mathrm{III}^{\mathrm{c}}&\le\frac{1}{2}\mathbb{E}\Big(\|\varepsilon_e^{n}\|_{\mathbb{L}^2([L_f,L_r])}\|\eta^{n+1}-\eta^{n}\|_{\mathbb{L}^2([L_f,L_r])}\|\Delta\tilde{W}_n\|_{\mathbb{L}^\infty([L_f,L_r])}\Big)\\&\le C\Delta t\mathbb{E}\|\eta^{n+1}-\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\Delta t^{-1}\mathbb{E}\Big(\|\varepsilon_e^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}\|\Delta\tilde{W}_n\|^2_{\mathbb{L}^\infty([L_f,L_r])}\Big)\\&\le C\Delta t\mathbb{E}\|\eta^{n+1}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\Delta t\mathbb{E}\|\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+Ch^{2k+2},\end{aligned}$$

where in the last inequalities for the estimate of $\mathrm{III}^{\mathrm{b}}$ and $\mathrm{III}^{\mathrm{c}}$, we use the independent property of Wiener process. The estimate of term *IV* is similar as that of term *III*, so we omit the process here.

Finally, V only contains flux difference terms which all vanish upon a summation in j. Combining these together, we know that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big(\|\varepsilon^{n+1}\|^2_{\mathbb{L}^2([L_f,L_r])}+\|\eta^{n+1}\|^2_{\mathbb{L}^2([L_f,L_r])}\Big)-\frac{1}{2}\mathbb{E}\Big(\|\varepsilon^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+\|\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}\Big)\\
&\le C\Delta t\mathbb{E}\|\varepsilon^{n+1}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\Delta t\mathbb{E}\|\varepsilon^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}\\
&\quad+C\Delta t\mathbb{E}\|\eta^{n+1}\|^2_{\mathbb{L}^2([L_f,L_r])}+C\Delta t\mathbb{E}\|\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+Ch^{2k+2}+C\Delta th^{2k+2}.
\end{aligned}$$

By Gronwall's inequality, there exists a constant $h_0>0$, for $h\le h_0$, we obtain

$$\mathbb{E}\Big(\|\varepsilon^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}+\|\eta^{n}\|^2_{\mathbb{L}^2([L_f,L_r])}\Big)\le Ch^{2k+2}+C\Delta t^{-1}h^{2k+2},\quad \forall n.$$

That is,

$$\mathbb{E}\|u^n-u_h^n\|^2_{\mathbb{L}^2}\le Ch^{2k+2}+C\Delta t^{-1}h^{2k+2}. \tag{3.28}$$

The proof is finished. □

3.3 *Main result*

Combining Theorems 3.3 and 3.4, we obtain the error estimate of (2.7).

THEOREM 3.5 Let $u(x,t)$ be the exact solution of the problem (1.1) and assume the initial value $u_0(x)\in L^2(\Omega;\mathbb{H}^{k+2})$ and $\phi\in\mathcal{L}_2(\mathbb{L}^2;\mathbb{H}^{k+2})$ $(k\ge 1)$. Let u_h^n be the numerical solution of the symplectic local discontinuous Galerkin method (2.7). Then there exists a constant $h_0>0$ such that for $h\le h_0$ holds

$$\mathbb{E}\|u(t_n)-u_h^n\|^2_{\mathbb{L}^2}\le C\Delta t^2+Ch^{2k+2}+C\Delta t^{-1}h^{2k+2}. \tag{3.29}$$

The overall convergence rate is usually expressed in terms of the computational cost of the scheme (Jentzen & Kloeden, 2011). Here the computational cost of method (2.7) is denoted by $M=N\cdot J$, with N and J being the total grid number in temporal and spacial directions, respectively. In view of the above error bound, it is optimal to choose $N=M^{\frac{2k+2}{2k+5}}$ and $J=M^{\frac{3}{2k+5}}$, i.e., $\Delta t=O\left(\frac{1}{N}\right)=O\left(\left(\frac{1}{M}\right)^{\frac{2k+2}{2k+5}}\right)$ and $h=O(\frac{1}{J})=O\left(\left(\frac{1}{M}\right)^{\frac{3}{2k+5}}\right)$, and we have the optimal error bound

$$\Big(\mathbb{E}\|u(t_n)-u_h^n\|^2_{\mathbb{L}^2}\Big)^{\frac{1}{2}}\le C\left(\frac{1}{M}\right)^{\frac{2k+2}{2k+5}}.$$

REMARK 3.6 If $k=1$, i.e., the initial data $u_0\in L^2(\Omega;\mathbb{H}^3)$ and $\phi\in\mathcal{L}_2(\mathbb{L}^2;\mathbb{H}^3)$, then the mean-square convergence rate of the method (2.7) with respect to the computational cost is $\frac{4}{7}$.

REMARK 3.7 In Section 3, the mean-square convergence was derived for the symplectic local discontinuous Galerkin method (2.7) discretized equation (1.1). Note that (1.1) is the linear Schrödinger equation.

As for nonlinear equation, truncation strategy may be needed to deal with the nonlinear term, as in De Bouard & Debussche (2004, 2006) and Liu (2013). However, things are a bit technical for the error estimation of the symplectic local discontinuous Galerkin method, since if we employ truncated strategy then it has to start by taking $\mathbb{H}^{\gamma}$-norm $\left(\gamma > \frac{d}{2}\right)$ on the error equation; see Remark 3.2 in Liu (2013). It looks like other technical strategy is needed to derive the mean-square convergence for symplectic local discontinuous Galerkin method applied to nonlinear case, and it will be our future work.

Funding

NNSFC (No. 91130003, No. 11021101, No. 11290142 and No. 11471310).

References

ANTONOPOULOU, D. C. & PLEXOUSAKIS, M. (2010) Discontinuous Galerkin methods for the linear Schrödinger equation in non-cylindrical domains. *Numer. Math.*, **115**, 585–608.

BANG, O., CHRISTIANSEN, P. L., IF, F., RASMUSSEN, K. O. & GAIDIDEI, YU. (1994) Temperature effects in a nonlinear model of monolayer Scheibe aggregates. *Phys. Rev. E*, **49**, 4627–4636.

BASS, F. G., KIVSHAR, Y. S., KONOTOP, V. V. & PRITULA, G. M. (1989) On stochastic dynamics of solitons in inhomogeneous optical fibers. *Opt. Commun.*, **70**, 309–314.

DE BOUARD, A. & DEBUSSCHE, A. (2003) The stochastic nonlinear Schrödinger equation in H^1. *Stoch. Anal. Appl.*, **21**, 97–126.

DE BOUARD, A. & DEBUSSCHE, A. (2004) A semi-discrete scheme for the stochastic nonlinear Schrödinger equation. *Numer. Math.*, **96**, 733–770.

DE BOUARD, A. & DEBUSSCHE, A. (2006) Weak and strong order of convergence of a semi-discrete scheme for the stochastic nonlinear Schrödinger equation. *Appl. Math. Optim.*, **54**, 369–399.

CHEN, C. & HONG, J. (2016) Symplectic Runge-Kutta semi-discretization for stochastic Schrödinger equation. *SIAM J. Numer. Anal.*, In Press.

CIARLET, P. G. (1978) *The Finite Element Method for Elliptic Problems*. Amsterdam: North-Holland.

COCKBURN, B., HOU, S. & SHU, C.-W. (2000) The development of discontinuous Galerkin methods, *Lecture Notes in Computational Science and Engineering* (B. Cockburn, G. Karniadakis & C.-W. Shu eds). Berlin: Springer, vol. 11, pp. 3–50.

COCKBURN, B. & SHU, C.-W. (1998) The local discontinuous Galerkin methods for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.*, **35**, 2440–2463.

COCKBURN, B. & SHU, C.-W. (2001) Runge-Kutta discontinuous Galerkin methods for convection-dominated problems. *J. Sci. Comput.*, **16**, 173–261.

ELGIN, J. (1993) Stochastic perturbations of optical solitons. *Phys. Lett. A*, **181**, 54–60.

JENTZEN, A. & KLOEDEN, P. E. (2011) Taylor approximations for stochastic partial differential equations. *SIAM.*, **83**.

LIU, J. (2013) Order of convergence of splitting schemes for both deterministic and stochastic nonlinear Schrödinger equations. *SIAM J. Numer. Anal.*, **51**, 1911–1932.

MILSTEIN, G. N., REPIN, YU., M. & TRETYAKOV, M. V. (2002) Numerical methods for stochastic systems preserving symplectic structure. *SIAM J. Numer. Anal.*, **40**, 1583–1604.

RASMUSSEN, K., GAIDIDEI, YU., BANG, O. & CHRISTIANSEN, P. (1995) The influence of noise on critical collapse in the nonlinear Schrödinger equation. *Phys. Lett. A*, **204**, 121–127.

XU, Y. & SHU, C.-W. (2005) Local discontinuous Galerkin methods for nonlinear Schrödinger equations. *J. Comput. Phys.*, **205**, 72–97.